Automated Market Making and Arbitrage Profits in the Presence of Fees

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Abstract. We consider the impact of trading fees on the profits of arbitrageurs trading against an automated marker marker (AMM) or, equivalently, on the adverse selection incurred by liquidity providers due to arbitrage. We extend the model of Milionis et al. [10] for a general class of two asset AMMs to both introduce fees and discrete Poisson block generation times. In our setting, we are able to compute the expected instantaneous rate of arbitrage profit in closed form.

Keywords: Blockchain · Decentralized Finance · Automated Market Makers.

1 Introduction

For automated market makers (AMMs), the primary cost incurred by liquidity providers (LPs) is adverse selection. Adverse selection arises from the fact that agents ("arbitrageurs") with an informational advantage, in the form of knowledge of current market prices, can exploit stale prices on the AMM versus prices on other markets such as centralized exchanges. Because trades between arbitrageurs and the AMM are zero sum, any arbitrage profits will be realized as losses to the AMM LPs. Milionis et al. [10] quantify these costs through a metric called loss-versus-rebalancing (LVR). They establish that LVR can be simultaneously interpreted as: (1) arbitrage profits due to stale AMM prices; (2) the loss incurred by LPs relative to a trading strategy (the "rebalancing strategy") that holds the same risky positions as the pool, but that trades at market prices rather than AMM prices; and (3) the value of the lost optionality when an LP commits upfront to a particular liquidity demand curve. They develop formulas for LVR in closed form, and show theoretically and empirically that, once market risk is hedged, the profit-and-loss (P&L) of an LP reduces to trading fee income minus LVR. In this way, LVR isolates the costs of liquidity provision.

Despite its benefits, LVR is derived under the simplification that arbitrageurs do not pay trading fees. In practice, however, trading fees pose a significant friction and limit arbitrage profits. Our goal is to introduce fees and understand how they impact arbitrageur behavior. As a starting point, one could directly introduce fees

into the model of Milionis et al. [10], where prices follow a geometric Brownian motion and arbitrageurs continuously monitor the AMM. However, this approach suffers a major pathology: when arbitrageurs monitor the market continuously in the presence of even negligible non-zero fees, the arbitrage profits are zero! Intuitively, when there are no fees, every instantaneous price movement provides a profitable arbitrage opportunity. With fees, this is true only for movements outside a (fee-dependent) "no-trade region" around the AMM price which, with continuous monitoring, then results in an immediate repositioning of that region. One can show that the fraction of time for which this happens is zero, with the market price inside the no-trade region at all other times. This is analogous to the fact that, in continuous time, a reflected random walk spends almost none of its time at the boundaries. In reality, however, arbitrageurs cannot continuously monitor and trade against the AMM. For example, for an AMM implemented on a blockchain, the arbitrageurs can only act at the discrete times at which blocks are generated. Thus, in order to understand arbitrage profits in the presence of fees, it is critical to model the discreteness of block generation.

1.1   Model

Our starting point is the model of Milionis et al. [10], where arbitrageurs continuously monitor an AMM to trade a risky asset versus the numéraire, and the risky asset price follows geometric Brownian motion parameterized by volatility $\sigma > 0$. However, we assume that the AMM has a trading fee $\gamma \geq 0$, and that arbitrageurs arrive to trade on the AMM at discrete times according to the arrivals of a Poisson process with rate $\lambda > 0$. The Poisson process is a natural choice because of its memoryless nature and standard usage throughout continuous time finance. It is natural to assume arrival times correspond to block generation times, since the arbitrageurs can only trade at instances where block are generated, so the parameter λ should be calibrated so that the mean interarrival time $\Delta t \triangleq \lambda^{-1}$ corresponds to the mean interblock time.

When an arbitrageur arrives, they seek to make a trade that myopically maximizes their immediate profit. Arbitrageurs trade myopically because of competition. If they choose to forgo immediate profit but instead wait for a larger mispricing, they risk losing the profitable trading opportunity to the next arbitrageur. If the AMM price net of fees is below (respectively, above) the market price, the arbitrageur will buy (sell) from the pool and sell (buy) at the market. They will do so until the net marginal price of the AMM equals the market price. We describe these dynamics in terms of a mispricing process that is the difference between the AMM and market log-prices. At each arrival time, a myopic arbitrageur will trade in a way such that the pool mispricing to jumps to the nearest point in band around zero mispricing. The width of the band is determined by the fee γ. We call this band the *no-trade region*, since if the arbitrageur arrives and the mispricing is already in the band, there is no profitable trade possible. At all non-arrival times, the mispricing is a diffusion, driven by the geometric Brownian motion governing market prices.

1.2 Results

In our setting, the mispricing process is a Markovian jump-diffusion process. Our first result is to establish that this process is ergodic, and to identify its steady state distribution in closed form. Under this distribution, the probability that an arbitrageur arrives and can make a profitable trade, i.e., the fraction of time that the mispricing process is outside the no-trade region in steady state, is given by

$$\mathsf{P}_{\mathsf{trade}} \triangleq \frac{1}{1 + \underbrace{\sqrt{2\lambda}\gamma/\sigma}_{\triangleq \eta}}.$$

This can also be interpreted as the long run fraction of blocks that contain an arbitrage trade. $\mathsf{P}_{\mathsf{trade}}$ has intuitive structure in that it is a function of the composite parameter $\eta \triangleq \gamma/(\sigma\sqrt{\lambda^{-1}/2})$, the fee measured as a multiple of the typical (one standard deviation) movement of returns over half the average interarrival time. When η is large (e.g., high fee, low volatility, or frequent blocks), the width of the no-fee region is large relative to typical interarrival price moves, so the mispricing process is less likely to exit the no-trade region in between arrivals, and $\mathsf{P}_{\mathsf{trade}} \approx \eta^{-1}$.

Given the steady state distribution of the pool mispricing, we can quantify the arbitrage profits. Denote by ARB_T the cumulative arbitrage profits over the time interval $[0, T]$. We compute the expected instantaneous rate of arbitrage profit $\overline{\mathsf{ARB}} \triangleq \lim_{T\to 0} \mathsf{E}[\mathsf{ARB}_T]/T$, where the expectation is over the steady state distribution of mispricing. We derive a semi-closed form expression (involving an expectation) for $\overline{\mathsf{ARB}}$. For specific cases, such as geometric mean or constant product market makers, this expectation can be evaluated resulting in an explicit closed form.

1.3 Conclusion

This work has broad implications around liquidity provision and the design of automated market makers. First, the model presented hereby provides a more accurate quantification of LP P&L, accounting both for arbitrageurs paying trading fees and discrete arbitrageur arrival times. As such, this model can be used for empirical analyses to evaluate LP performance both *ex post* as well as *ex ante*, when coupled with realized metrics of pool data, such as realized asset price volatility. Our results also have the potential to better inform AMM design, and in particular, provide guidance around how to set trading fees in a competitive LP market, in order to balance LP fee income and LP loss due to arbitrageurs.

1.4 Literature Review

There is a rich literature on automated market makers. Angeris and Chitra [1] and Angeris et al. [2, 3] apply tools from convex analysis (e.g., the pool reserve value function) that we also use in this paper. In the first paper to decompose the

return of an LP into an instantaneous market risk component and a non-negative, non-decreasing, and predictable component called "loss-versus-rebalancing" (LVR, pronounced "lever"), Milionis et al. [10] analyze the frictionless, continuous-time Black-Scholes setting in the absence of trading fees to show that it is exactly the adverse selection cost due to the arbitrageurs' informational advantage to the pool. This work extends the model of Milionis et al. [10] to account for arbitrage profits both in the presence of fees and discrete-time arbitrageur arrivals. Broader classes of AMMs that have locally smooth demand curves but are not necessarily constant function market makers have been given by Milionis et al. [8, 9]; our model here applies to such a general case as well. Evans et al. [7] observe that, in the special case of geometric mean market makers, taking the limit to continuous time while holding the fees $\gamma > 0$ fixed and strictly positive yields vanishing arbitrage profits; this is also a special case of our results. Angeris et al. [3] also analyze arbitrage profits, but do not otherwise express them in closed-form. Black-Scholes-style options pricing models, like the ones developed in this paper, have been applied to weighted geometric mean market makers over a finite time horizon by Evans [6], who also observes that constant product pool values are a super-martingale because of negative convexity. Clark [4] replicates the payoff of a constant product market over a finite time horizon in terms of a static portfolio of European put and call options. Tassy and White [13] compute the growth rate of a constant product market maker with fees. Dewey and Newbold [5] develop a model of pricing and hedging AMMs with arbitrageurs and noise traders and conjecture that arbitrageurs induce the same stationary distribution of mispricing that we rigorously develop here.

2   Model

Assets. Fix a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}\big)$ satisfying the usual assumptions. Consider two assets: a risky asset x and a numéraire asset y. Working over continuous times $t \in \mathbb{R}_+$, assume that there is observable external market price P_t at each time t. The price P_t evolves exogenously according to the geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu\, dt + \sigma\, dB_t, \quad \forall\, t \geq 0,$$

with drift μ, volatility $\sigma > 0$, and where B_t is a Brownian motion.

AMM Pool. We assume that the AMM operates as a constant function market maker (CFMM). The state of a CFMM pool is characterized by the reserves $(x, y) \in \mathbb{R}_+^2$, which describe the current holdings of the pool in terms of the risky asset and the numéraire, respectively. Define the feasible set of reserves $\mathcal{C}$ according to

$$\mathcal{C} \triangleq \{(x, y) \in \mathbb{R}_+^2 \ : \ f(x, y) = L\},$$

where $f \colon \mathbb{R}_+^2 \to \mathbb{R}$ is referred to as the *bonding function* or *invariant*, and $L \in \mathbb{R}$ is a constant. In other words, the feasible set is a level set of the bonding function.

The pool is defined by a smart contract which allows an agent to transition the pool reserves from the current state $(x_0, y_0) \in \mathcal{C}$ to any other point $(x_1, y_1) \in \mathcal{C}$ in the feasible set, so long as the agent contributes the difference $(x_1 - x_0, y_1 - y_0)$ into the pool. Define the *pool value function* $V \colon \mathbb{R}_+ \to \mathbb{R}_+$ by the optimization

$$
V(P) \triangleq \operatorname*{minimize}_{(x,y) \in \mathbb{R}_+^2} \quad Px + y \\
\text{subject to } f(x,y) = L.
\tag{1}
$$

The pool value function yields the value of the pool, assuming that the external market price of the risky asset is given by P, and that arbitrageurs can trade instantaneously trade against the pool maximizing their profits (and simultaneously minimizing the value of the pool). Geometrically, the pool value function implicitly defines a reparameterization of the pool state from primal coordinates (reserves) to dual coordinates (prices).

Following Milionis et al. [10], we assume that the pool value function satisfies:

Assumption 1 *(i) An optimal solution $\big(x^*(P), y^*(P)\big)$ to the pool value optimization (1) exists for every $P \geq 0$.*
(ii) The pool value function $V(\cdot)$ is everywhere twice continuously differentiable.
(iii) For all $t \geq 0$,

$$
\mathsf{E}\left[\int_0^t x^*(P_s)^2 P_s^2 \, ds\right] < \infty.
$$

We refer to $\big(x^*(P), y^*(P)\big)$ as the *demand curves* of the pool for the risky asset and numéraire, respectively. Assumption 1(i)–(ii) is a sufficient condition for the following:

Lemma 1. *For all prices $P \geq 0$, the pool value function satisfies:*

(i) $V(P) \geq 0$.
(ii) $V'(P) = x^*(P) \geq 0$.
(iii) $V''(P) = x^{*\prime}(P) = -Py^{*\prime}(P) \leq 0$.

The proof of Lemma 1 follows from standard arguments in convex analysis; see Milionis et al. [10] for details.

Fee Structure. Suppose that $(\Delta x, \Delta y)$ is a feasible trade permitted by the pool invariant, i.e., given initial pool reserves (x, y) with $f(x, y) = L$, we have $f(x + \Delta x, y + \Delta y) = L$. We assume that an additional proportional trading fee is paid to the LPs in the pool. The mechanics of this trading fee are as follows:

1. The fee is paid in the numéraire in proportion to the quantity Δy of numéraire that is traded.
2. The fee is realized as a separate cashflow to the LPs.
3. We allow for different fees to be paid when the risky asset is bought from the pool and when the risky asset is sold to the pool.

4. We denote the fee in units of log price by $\gamma_+, \gamma_- > 0$. In particular, when the agent purchases the risky asset from the pool (i.e., $\Delta x < 0, \Delta y > 0$), the total fee charged is

$$\left(e^{+\gamma_+} - 1\right)|\Delta y|, \tag{2}$$

while the total fee charged when the agent sells the risky asset to the pool (i.e., $\Delta x > 0, \Delta y < 0$ is

$$\left(1 - e^{-\gamma_-}\right)|\Delta y|. \tag{3}$$

Example 1. In our notation, a 30 basis point proportional fee on either buys or sales (e.g., as in Uniswap V2) would correspond to

$$\gamma_+ = \log(1 + 0.003) \approx 0.002995509, \quad \gamma_- = -\log(1 - 0.003) \approx 0.003004509.$$

To a first order, $\gamma_+ \approx \gamma_- \approx 30$ (basis points).

3   Arbitrageurs & Pool Dynamics

At any time $t \geq 0$, define $\tilde{P}_t$ to be the price of the risky asset implied by pool reserves, i.e., the reserves are be given by $\left(x^*(\tilde{P}_t), y^*(\tilde{P}_t)\right)$. Denote by

$$z_t \triangleq \log P_t/\tilde{P}_t, \tag{4}$$

the log mispricing of the pool, so that $\tilde{P}_t = P_t e^{-z_t}$.

We imagine that arbitrageurs arrive to trade against the pool at discrete times according to a Poisson process of rate $\lambda > 0$. Here, we imagine that arbitrageurs are continuously monitoring the market, but can only trade against the pool at discrete times when blocks are generated in a blockchain. Hence, we will view the arrival process as both equivalently describing the arrival of arbitrageurs to trade or times of block generation. For a proof-of-work blockchain, Poisson block generation is a natural assumption [11]. However, modern proof-of-state blockchains typically generate blocks at deterministic times. In these cases, we will view the Poisson assumption as an approximation that is necessary for tractability.

Denote the arbitrageur arrival times (or block generation times) by $0 < \tau_1 < \tau_2 < \cdots$. When an arbitrageur arrives at time $t = \tau_i$, they can trade against the pool (paying the relevant trading fees) according to the pool mechanism, and simultaneously, frictionlessly trade on an external market at the price P_t. We assume that the arbitrageur will trade to myopically maximize their instantaneous trading profit.[4] We ignore any blockchain transaction fees such as "gas".

The following lemma characterizes the myopic behavior of the arbitrageurs in terms of the demand curves of the pool and the fee structure:

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[4] Given trading fees, if there was a single, monopolist arbitrageur, this may not be optimal, e.g., it may be optimal to wait for a large mispricing before trading. However, we assume that there exists a universe of competing arbitrageurs, and that an arbitrageur that forgoes any immediate profit will lose it to a competitor. Hence, in our setting, arbitrageurs trade myopically.

Lemma 2. *Suppose that an arbitrageur arrives at time $t = \tau_i$, observing external market price P_t, and implied pool price $\tilde{P}_{t-}$ or, equivalently, mispricing z_{t-}. Then, one of the following three cases applies:*

1. *If $P_t > \tilde{P}_{t-}e^{+\gamma_+}$ or, equivalently, $z_{t-} > +\gamma_+$, the arbitrageur can profitably buy in the pool and sell on the external market. They will do so until the pool price satisfies $\tilde{P}_t = P_t e^{-\gamma_+}$ or, equivalently, $z_t = +\gamma_+$. The arbitrageur profits are then*

$$P_t \left\{ x^* \left(P_t e^{-z_{t-}} \right) - x^* \left(P_t e^{-\gamma_+} \right) \right\} + e^{+\gamma_+} \left\{ y^* \left(P_t e^{-z_{t-}} \right) - y^* \left(P_t e^{-\gamma_+} \right) \right\} \geq 0.$$

2. *If $P_t < \tilde{P}_{t-}e^{-\gamma_-}$ or, equivalently, $z_{t-} < -\gamma_-$, the arbitrageur can profitably sell in the pool and buy the external market. The will do so until the pool price satisfies $\tilde{P}_t = P_t e^{+\gamma_-}$ or, equivalently, $z_t = -\gamma_-$. The arbitrageur profits are then*

$$P_t \left\{ x^* \left(P_t e^{-z_{t-}} \right) - x^* \left(P_t e^{+\gamma_-} \right) \right\} + e^{-\gamma_-} \left\{ y^* \left(P_t e^{-z_{t-}} \right) - y^* \left(P_t e^{+\gamma_-} \right) \right\} \geq 0.$$

3. *If $\tilde{P}_{t-}e^{-\gamma_-} \leq P_t \leq \tilde{P}_{t-}e^{+\gamma_+}$, then the arbitrageur makes no trade, and $\tilde{P}_t = \tilde{P}_{t-}$ or, equivalently, $z_t = z_{t-}$.*

Considering the three cases in Lemma 2, it is easy to see that, at an arbitrageur arrival time τ_i, the mispricing process z_t evolves according to[5]

$$z_{\tau_i} = \text{bound} \left\{ z_{\tau_i^-}, -\gamma_-, +\gamma_+ \right\}, \tag{5}$$

On the other hand, applying Itô's lemma to (4), we have that, at other times $t > 0$, process evolves according to
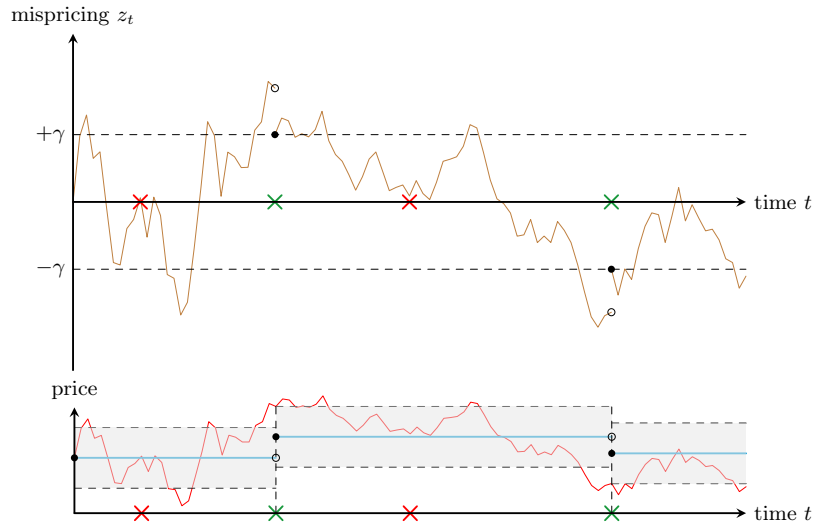
$$dz_t = \left(\mu - \tfrac{1}{2}\sigma^2 \right) dt + \sigma \, dB_t. \tag{6}$$

Combining (5)–(6), for all $t \geq 0$,

$$z_t = \left(\mu - \tfrac{1}{2}\sigma^2 \right) t + \sigma B_t + \sum_{i: \, \tau_i \leq t} J_i, \qquad J_i \triangleq \text{bound} \left\{ z_{\tau_i^-}, -\gamma_-, +\gamma_+ \right\} - z_{\tau_i^-}. \tag{7}$$

Therefore, the mispricing process z_t is a Markovian jump-diffusion process. Possible sample paths of all of these processes are shown in Figure 1.

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[5] Define $\text{bound}\{x, u, \ell\} \triangleq \max(\min(x, u), \ell)$.

Fig. 1. Top: example sample path of the mispricing process z_t. Bottom: in red, example external market price P_t; in blue, example implied pool price $\tilde{P}_{t^-}$. The no-trade interval is shown in shaded gray; whenever the external market price is within this interval, no trade will happen even if an arbitrageur arrival occurs. The red- and green-colored crosses in the x-axis show the (Poisson-distributed) times of arbitrageur arrivals; the red ones show the arrivals on which arbitrageurs do not trade with the pool, while the green ones correspond to the arrivals on which arbitrageurs do trade.

4   Exact Analysis

Assumption 2 (Symmetry) $\mu = \frac{1}{2}\sigma^2$, $\gamma_+ = \gamma_- \triangleq \gamma$.

Assumption 2 ensures that the mispricing jump-diffusion process, with dynamics given by (5)–(6), is driftless and is distributed symmetrically around the $z = 0$ axis.

4.1   Stationary Distribution of the Mispricing Process

The following lemma characterizes the stationary distribution of the mispricing process.[6]

Theorem 1 (Stationary Distribution of Mispricing).   *The process z_t is an ergodic process on $\mathbb{R}$, with unique invariant distribution $\pi(\cdot)$ given by the density*

$$
p_\pi(z) = \begin{cases} \pi_+ \times p_{\eta/\gamma}^{\exp}(z - \gamma) & \text{if } z > +\gamma, \\ \pi_0 \times \frac{1}{2\gamma} & \text{if } z \in [-\gamma, +\gamma], \\ \pi_- \times p_{\eta/\gamma}^{\exp}(-\gamma - z) & \text{if } z < -\gamma, \end{cases}
$$

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[6] Contemporaneous with the present work, Dewey and Newbold [5] conjecture this stationary distribution.

for $z \in \mathbb{R}$. Here, we define the composite parameter $\eta \triangleq \sqrt{2\lambda}\gamma/\sigma$. The probabilities π_-, π_0, π_+ of the three segments are given by

$$\pi_0 \triangleq \pi\big([-\gamma, +\gamma]\big) = \frac{\eta}{1+\eta}, \quad \pi_+ \triangleq \pi\big((+\gamma, +\infty)\big) = \pi_- \triangleq \pi\big((-\infty, -\gamma)\big) = \frac{1}{2}\frac{1}{1+\eta}.$$

*Finally, $p_{\eta/\gamma}^{\exp}(x) \triangleq (\eta/\gamma)e^{-(\eta/\gamma)x}$ is the density of an exponential distribution over $x \geq 0$ with parameter $\eta/\gamma = \sqrt{2\lambda}/\sigma$.*

The stationary distribution is illustrated in Figure 2.

Under this distribution, the probability that an arbitrageur arrives and can make a profitable trade, i.e., the fraction of time that the mispricing process is outside the no-trade region in steady state, is given by

$$\mathsf{P}_{\text{trade}} \triangleq \pi_+ + \pi_- = \frac{1}{1 + \sqrt{2\lambda}\gamma/\sigma}.$$

Equivalently, $\mathsf{P}_{\text{trade}}$ can be interpreted as the long run fraction of blocks that contain an arbitrage trade.
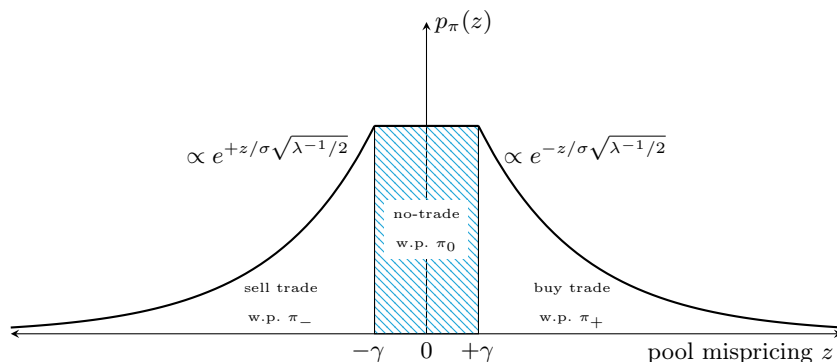

Fig. 2. The density $p_\pi(z)$ of the stationary distribution $\pi(\cdot)$ of mispricing z, illustrating trade and no-trade regions for an arbitrageur.

Note that $\mathsf{P}_{\text{trade}}$ *does not depend* on the bonding function or feasible set defining the CFMM pool; the only pool property relevant is the fee γ. $\mathsf{P}_{\text{trade}}$ has intuitive structure in that it is a function of the composite parameter $\eta \triangleq \gamma/(\sigma\sqrt{\lambda^{-1}/2})$, the fee measured as a multiple of the typical (one standard deviation) movement of returns over half the average interarrival time. When η is large (e.g., high fee, low volatility, or frequent blocks), the width of the no-fee region is large relative to typical interarrival price moves, so the mispricing process is less likely to exit the no-trade region in between arrivals, and $\mathsf{P}_{\text{trade}} \approx \eta^{-1}$. Example calculations of $\mathsf{P}_{\text{trade}}$ are shown in Table 1 for $\sigma = 5\%$ (daily) volatility and varying mean interblock times $\Delta t \triangleq \lambda^{-1}$ and fee levels γ, as well as in Figure 3a.

$\Delta t \setminus \gamma$	1 bp	5 bp	10 bp	30 bp	100 bp
10 min	96.7%	85.5%	74.7%	49.6%	22.8%
2 min	92.9%	72.5%	56.9%	30.5%	11.6%
12 sec	80.7%	45.6%	29.5%	12.3%	4.0%
2 sec	63.0%	25.4%	14.5%	5.4%	1.7%
50 msec	21.2%	5.1%	2.6%	0.9%	0.3%

Table 1. The probability of trade $\mathsf{P}_{\text{trade}}$, or, equivalently, the fraction of blocks containing an arbitrage trade, given asset price volatility $\sigma = 5\%$ (daily), with varying mean interblock times $\Delta t \triangleq \lambda^{-1}$ and fee levels γ (in basis points).
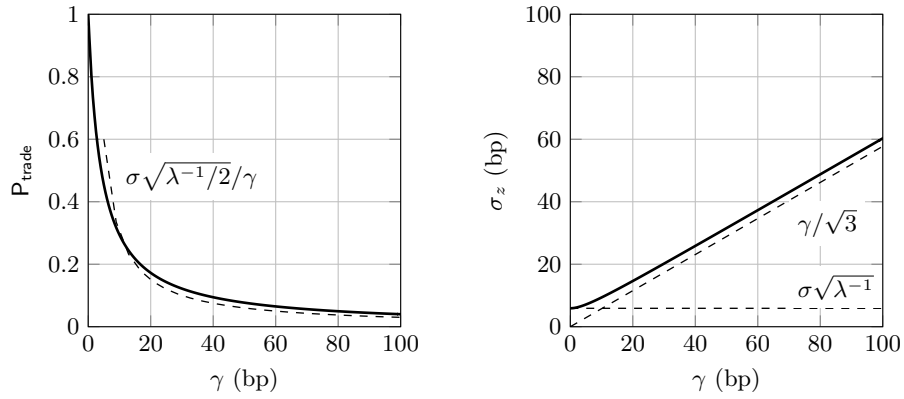
(a) The probability of trade $\mathsf{P}_{\text{trade}}$, or, equivalently, the fraction of blocks containing an arbitrage trade, as a function of the fee γ.

(b) The standard deviation of mispricing σ_z, as a function of the fee γ.

Fig. 3. Probability of trade and typical mispricing errors as a function of the fee, with $\sigma = 5\%$ (daily) and mean interblock time $\Delta t \triangleq \lambda^{-1} = 12$ (seconds).

The following immediate corollary quantifies the magnitude of a typical mispricing. This is illustrated in Figure 3b.

Corollary 1 (Standard Deviation of Mispricing). *Under the invariant distribution $\pi(\cdot)$, the standard deviation of the mispricing is given by*

$$\sigma_z \triangleq \sqrt{\mathsf{E}_\pi[z^2]} = \sqrt{(1 - \mathsf{P}_{\text{trade}}) \times \tfrac{1}{3}\gamma^2 + \mathsf{P}_{\text{trade}} \times \left\{\left(\gamma + \frac{\sigma}{\sqrt{2\lambda}}\right)^2 + \frac{\sigma^2}{2\lambda}\right\}}.$$

4.2   Rate of Arbitrageur Profit

Denote by N_T the total number of arbitrageur arrivals in $[0, T]$. Suppose an arbitrageur arrives at time τ_i, observing external price P_{τ_i} and mispricing $z_{\tau_i^-}$. From Lemma 2, the arbitrageur profit is given by

$$A(P_{\tau_i}, z_{\tau_i^-}) \triangleq A_+(P_{\tau_i}, z_{\tau_i^-}) + A_-(P_{\tau_i}, z_{\tau_i^-}) \geq 0,$$

where we define

$$A_+(P, z) \triangleq \left[P\left\{x^*\left(Pe^{-z}\right) - x^*\left(Pe^{-\gamma}\right)\right\}\right.$$
$$\left. + e^{+\gamma}\left\{y^*\left(Pe^{-z}\right) - y^*\left(Pe^{-\gamma}\right)\right\}\right]\mathbb{I}_{\{z > +\gamma\}} \geq 0,$$

$$A_-(P, z) \triangleq \left[P\left\{x^*\left(Pe^{-z}\right) - x^*\left(Pe^{+\gamma}\right)\right\}\right.$$
$$\left. + e^{-\gamma}\left\{y^*\left(Pe^{-z}\right) - y^*\left(Pe^{+\gamma}\right)\right\}\right]\mathbb{I}_{\{z < -\gamma\}} \geq 0.$$

Similarly, the fees paid by the arbitrageur in this scenarios is given by

$$F(P_{\tau_i}, z_{\tau_i^-}) \triangleq F_+(P_{\tau_i}, z_{\tau_i^-}) + F_-(P_{\tau_i}, z_{\tau_i^-}) \geq 0,$$

where we define

$$F_+(P, z) \triangleq -\left(e^{+\gamma} - 1\right)\left[y^*\left(Pe^{-z}\right) - y^*\left(Pe^{-\gamma}\right)\right]\mathbb{I}_{\{z > +\gamma\}} \geq 0,$$

$$F_-(P, z) \triangleq +\left(1 - e^{-\gamma}\right)\left[y^*\left(Pe^{-z}\right) - y^*\left(Pe^{+\gamma}\right)\right]\mathbb{I}_{\{z < -\gamma\}} \geq 0.$$

We can write the total arbitrage profit and fees paid over $[0, T]$ by summing over all arbitrageurs arriving in that interval, i.e.,

$$\mathsf{ARB}_T \triangleq \sum_{i=1}^{N_T} A(P_{\tau_i}, z_{\tau_i^-}), \quad \mathsf{FEE}_T \triangleq \sum_{i=1}^{N_T} F(P_{\tau_i}, z_{\tau_i^-}).$$

Clearly these are non-negative and monotonically increasing processes. The following theorem characterizes their instantaneous expected rate of growth or intensity:[7]

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[7] Mathematically, $\overline{\mathsf{ARB}}$ is the intensity of the compensator for the monotonically increasing jump process ARB_T at time $T = 0$, similarly $\overline{\mathsf{FEE}}$ is the intensity of the compensator for FEE_T.

Theorem 2 (Rate of Arbitrage Profit and Fees). *Define the intensity, or instantaneous rate of arbitrage profit, by $\overline{\mathsf{ARB}} \triangleq \lim_{T\to 0} \frac{\mathsf{E}[\mathsf{ARB}_T]}{T}$. Given initial price $P_0 = P$, suppose that $z_0 = z$ is distributed according to its stationary distribution $\pi(\cdot)$. Then, the instantaneous rate of arbitrage profit is given by*

$$\overline{\mathsf{ARB}} = \lambda \mathsf{E}_\pi \left[A(P,z) \right] = \lambda \mathsf{P}_{\mathsf{trade}} \frac{\sqrt{2\lambda}}{\sigma} \int_0^\infty \frac{A_+(P, x+\gamma) + A_-(P, -x-\gamma)}{2} e^{-\sqrt{2\lambda}x/\sigma} \, dx.$$

$$(8)$$

Similarly, defining the intensity of the fee process by $\overline{\mathsf{FEE}} \triangleq \lim_{T\to 0} \frac{\mathsf{E}[\mathsf{FEE}_T]}{T}$,

$$\overline{\mathsf{FEE}} = \lambda \mathsf{E}_\pi \left[F(P,z) \right] = \lambda \mathsf{P}_{\mathsf{trade}} \frac{\sqrt{2\lambda}}{\sigma} \int_0^\infty \frac{F_+(P, x+\gamma) + F_-(P, -x-\gamma)}{2} e^{-\sqrt{2\lambda}x/\sigma} \, dx.$$

$$(9)$$

4.3 Example: Constant Product Market Maker

Theorem 2 provides an exact, semi-analytic closed form expression for the rate of arbitrage profit, in terms of a certain Laplace transfrom of the functions $\{A_\pm(P,\cdot)\}$. This expression can be evaluated as an explicit closed form for many CFMMs. For example, consider the case of constant product market makers:

Corollary 2. *Consider a constant product market maker, with invariant $f(x,y) \triangleq \sqrt{xy} = L$. Under the assumptions of Theorem 2, the intensity per dollar value in the pool is given by[8]*

$$\frac{\overline{\mathsf{ARB}}}{V(P)} = \begin{cases} \dfrac{\sigma^2}{8} \times \mathsf{P}_{\mathsf{trade}} \times \dfrac{e^{+\gamma/2} + e^{-\gamma/2}}{2\left(1 - \sigma^2/(8\lambda)\right)} & \text{if } \sigma^2/8 < \lambda, \\[4mm] +\infty & \text{otherwise.} \end{cases}$$

where the quantities on the right side do not depend on the value of $P_0 = P$.

Under the normalization of Corollary 2, where the intensity of arbitrage profits is normalized relative the pool value, the resulting quantity does not depend on the price. The same property will hold for the more general class of geometric mean market makers; this is analogous to the property that LVR is proportional to pool value for this class [10].

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[8] Note that there are infinite expected arbitrage profits if $\lambda < \sigma^2/8$. This is a consequence of the interaction of the lognormal returns and the exponential interblock time. When blocks arrive very slowly, the interblock return can have large tails. This regime is not practically relevant, however. In particular, if $\sigma = 5\%$ (daily), then this occurs when the mean interblock time satisfies $\Delta t \triangleq \lambda^{-1} > 8/\sigma^2 = 3200$ (days).

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